Dividing both members of the last equation by θ,

$$i\left(\frac{\sin\theta}{\theta}\right) = k + \frac{k^3\theta^2}{\underline{|3}} + \frac{k^5\theta^4}{\underline{|5}} \cdots.$$

As $\theta \doteq 0$, $(\sin\theta)/\theta \doteq 1$, and since k is constant,

$$k = i.$$

Substitution of i for k in the various formulas cited affords series development of $\cos\theta$ and $i\sin\theta$, in the first of which all terms are real, and in the second all imaginary; and also establishes the notation

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

9. If at this stage *and not sooner* the vector is defined the student will grasp perfectly the idea that such a directed line has two values, an algebraic value and an absolute value, which are different, in that its algebraic value depends upon its direction determined by the length of the arc of a unit circle described about the origin as a center.

It is the experience of the writer that in secondary school classes this development requires about as many days as he has topics outlined above, and that certain difficulties relating to the *value* of the ordinate are thereby avoided. A favorable opening is also provided for the teaching of hyperbolic functions, when this is desirable.

III. On Proofs by Mathematical Induction.

By E. T. Bell, University of Washington.

1. The best way to cure oneself of a crotchet is to confide it to some sympathetic listener. The crotchet in this note is one which has worried me since school days when I was induced to repeat the proof of the binomial theorem by mathematical induction. The same crotchet seems to trouble successive generations of freshmen, for occasionally one has obstinacy enough to balk at the magic formula "*and therefore the theorem is always true,*" with which many authors conclude their proofs by recurrence. I hold that mathematical induction has no place in elementary teaching, particularly when such teaching strains at mathematical gnats, as in pseudo-rigorous presentations of the elementary theory of limits, the better to swallow logical camels such as some proofs of the binomial theorem or their equivalent quoted presently from Poincaré. This is the crotchet. In short, elementary teaching would be more convincing if it left rigor to that logistics which was Poincaré's bête noir.

2. It would be difficult to find a balder statement of the logical vice which characterizes many proofs by mathematical induction in the current text books, than the following extract from Poincaré's essay *On the Nature of Mathematical Reasoning*, in *Science and Hypothesis* (Halsted's translation, page 36, section IV).

Having considered three examples of "proof by recurrence," and having drawn a false conclusion in each of them, Poincaré says:

"Here I stop this monotonous series of reasonings. But this very monotony has the better brought out the procedure which is uniform and is met again at each step.

"This procedure is the demonstration by recurrence. We first establish a theorem for $n = 1$; then we show that if it is true of $n - 1$, it is true of n, and thence conclude that it is true for all whole numbers."

It is only fair to state that in the next section (V) of his essay, Poincaré gives an unobjectionable form of "this procedure," which he calls "mathematical reasoning *par excellence*." This, however, may have been an inadvertence, as the closing paragraph of the essay again exhibits the circularity of the reasoning in all its viciousness:

"Observe finally that this induction is possible only if the same operation can be repeated indefinitely."

From this we should expect a rich crop of subtle fallacies when "this induction" is applied to prove that a certain assemblage contains an infinity of members, or when it is used to demonstrate the universal truth of a proposition.

3. To exhibit the logical defect in this "mathematical reasoning par excellence," let us separate Poincaré's summary into its three constituents:

(1) "We first establish a theorem for $n = 1$;

(2) "Then we show that if it is true of $n - 1$, it is true of n;

(3) "And thence (we) conclude that it is true for all whole numbers."

In (3) "thence," if it means anything definite, must refer to (2) and (1). That is, the conclusion that the theorem is true for *all* whole numbers is to follow from (1) and (2) *only*. It does not so follow. In order to draw the conclusion (3), we need, either as a postulate, or, if it can be proved from simpler assumptions, the proposition:

(4) If a theorem is true for $n = 1$, and if its truth for $n - 1$ implies that it is true for n, then the theorem is true for all whole numbers.

Without (4), all that (1) and (2) give is the means for *step by step* assertions that the theorem is true in successive cases. Thus, supposing (1) and (2) established, if it be required to see whether the theorem is true for $n = 5$, we must, for all that (1) and (2) prove, take the steps 1 to 2, 2 to 3, 3 to 4, 4 to 5, omitting none. Or, again, if (1) and (2) are established, and we wish to know whether the theorem is true for $n = 9^{9^9}$, say, we must take $9^{9^9} - 1$ steps in order to find out, for neither (1) nor (2) permits us to take more than one step at a time. In this case we might never gratify our curiosity, and the truth or falsity of the theorem for $n = 9^{9^9}$ would remain as inaccessible to our knowledge as is the other side of the moon. Nor could we assert that the theorem *most probably* is true for this value of n, for we cannot predicate probabilities in the absence of data. Our belief that the theorem is true for this value of n might be strong; but belief belongs to the realm of emotional experiences, and is seldom conspicuous for the reasonableness of its tenets. Until (4) or its equivalent is proved or admitted as a postulate, it would seem to be advisable to banish such phrases as "all whole numbers," and "the same operation can be repeated indefinitely," from elementary texts which make pretensions to rigor.

4. If reasoning by recurrence is, as Poincaré claims, mathematical reasoning *par excellence*, and if the objections put forth in § 3 are not groundless, it would seem to follow that mathematics is like any other science in that the conclusions which it legitimately draws are no more "general" or "universal" than those of other sciences. This contradicts what seems to be a current valuation of mathematical truth in the minds of laymen and some others who hold that mathematics has a timeless, eternal aspect, independent of all the empiricism which characterizes the conclusions of physical sciences.

There is one way of escape which is so obvious that it need only be pointed out. We can beat the mathematical devil round the logical bush by saying that (4) of § 3 is the rule, or law, of inference. But it would be a wise logician indeed who recognized (4) as one of his legitimate children. For where is either a proof of it or its explicit statement as a postulate of logic to be found?

IV. A Practical Printer's Problem in Maxima and Minima.

By Edgar E. DeCou, University of Oregon.

Dean Eric W. Allen, of the School of Journalism of the University of Oregon, presents a very interesting problem of frequent occurrence to the practical printer. The printer's only method of solution is by trial and error; and he states that on a large job of printing an added cost of \$100 or \$200 is often incurred by inability to solve the problem.

The conditions of the problem are as follows: 200,000 (P) prints are required; 1200 (S) prints per hour is the speed of the press; \$2.00 (R) per hour is the cost of running the press; 55 cents (E) each is the cost of the extra electrotypes, needed after the type is once set up. Required the number of electrotypes (x) that should be used to secure the minimum cost (C).

The problem is evidently one in determining the minimum value of C by the use of the differential calculus. The particular case takes the form,

$$C \text{ (in cents)} = \frac{200{,}000 \times 200}{1200(1+x)} + 55x = \frac{100{,}000}{3(x+1)} + 55x,$$

where x represents the number of electrotypes. Differentiating

$$\frac{dC}{dx} = -\frac{100{,}000}{3} \cdot \frac{1}{(x+1)^2} + 55 = 0,$$

for minimum value of C. Hence

$$x = \frac{100}{33} \cdot \sqrt{66} - 1 = 23.6 + .$$